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ANY BANACH SPACE HAS AN EQUIVALENT NORM WITH TRIVIAL ISOMETRIES

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ABSTRACT

For any Banach space X there is a norm $|||\cdot|||$ on X, equivalent to the original one, such that $(X, |||\cdot|||)$ has only trivial isometries. For any group G there is a Banach space X such that the group of isometries of X is isomorphic to $G \times \{-1, 1\}$. For any countable group G there is a norm $\|\cdot\|_G$ on $C([0, 1])$ equivalent to the original one such that the group of isometries of $(C([0, 1]), \|\cdot\|_G)$ is isomorphic to $G \times \{-1, +1\}$.

§0. Introduction

In this note we prove that any Banach space $(X, \|\cdot\|)$ has an equivalent norm $|||\cdot|||$ such that $T : (X, |||\cdot|||) \to (X, |||\cdot|||)$ is a surjective linear isometry only if $T = \lambda \operatorname{Id}_X$ with $|\lambda| = 1$. This result holds in both the cases real and complex which are not equivalent.

Of course any complex Banach space is also a real Banach space and any complex-linear isometry is also a real-linear but a functional which is a norm over the reals need not be a norm over complex numbers.

A similar result for real separable spaces was obtained recently by S. F. Bellenot [1]. The method of our proof is, roughly speaking, to construct, for a given Banach space $(X, \|\cdot\|)$, a seminorm $p(\cdot)$ on X such that only maps of the form $\lambda \operatorname{Id}_X$, $|\lambda| = 1$ preserve both $\|\cdot\|$ and $p(\cdot)$. Then we construct a norm $|||\cdot|||$ on $X \oplus K$ where $\dim K = 1$, with the property that $T : (X, |||\cdot|||) \to (X, |||\cdot|||)$ is an isometry if and only if it is both $\|\cdot\|$- and $p(\cdot)$-isometry.

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For a Banach space X we call a group G representable in X if there is an equivalent norm $|||\cdot|||$ on X such that G and the group of all surjective linear isometries of $(X, |||\cdot|||)$ are isomorphic. It seems to be an open problem to describe for a given Banach (Hilbert) space X the family of all groups representable in X. Evidently not all groups are representable, e.g. the one point group. However, we observe as an easy consequence of the result of J. De Groot [3] and V. Kannan and M. Rajagopalan [6] that for any group G there is a Banach space X such that $G \times \{-1, 1\}$ is representable in X.

§1. Definitions and notation

In the paper we use the standard Banach space terminology. For a Banach space E, by E_1 we denote the closed unit ball of E and by ∂E_1 the set $\{e \in E : \| e \| = 1\}$. By K we denote both the sets of real and complex numbers and so $K_1 = \{\lambda \in K : |\lambda| \leqq 1\}$. By an isometry on E we mean a surjective linear map $T : E \to E$ such that $\| Te \| = \| e \|$ for any $e \in E$. If we have more than one norm on E we call such a map $\|\cdot\|$-isometry and we write $(E, \|\cdot\|)$ in place of E, to indicate to which norm we refer. By a trivial isometry we mean any map of the form $\lambda \operatorname{Id}_E$, $|\lambda| = 1$.

For a set Γ we denote by $l^\infty(\Gamma)$ the Banach space of all bounded scalar valued functions on Γ with the obvious sup norm. For any $\gamma \in \Gamma$, $e_\gamma \in l^\infty(\Gamma)$ is the characteristic function of the set $\{\gamma\}$ and e_γ^* the functional on $l^\infty(E)$-value at the point γ. We denote by $c_0(\Gamma)$ the smallest closed subspace of $l^\infty(\Gamma)$ which contains all e_γ for $\gamma \in \Gamma$.

For Banach spaces X, Y we denote by $X \oplus Y$ the direct sum of X and Y.

For a compact, Hausdorff space S, $C_{\mathbf{R}}(S)$ (respectively $C_{\mathbf{C}}(S)$) is the Banach space of all real (respectively complex) continuous functions defined on S, with the obvious sup norm.

For groups G_1 and G_2, by $G_1 \times G_2$ we denote the product of G_1 and G_2 with the obvious group operation. We consider ∂K_1, which is $\{-1, +1\}$ or the unit circle, as a multiplicative group.

§2. The results

THEOREM 1. *For any Banach space X there is a Banach space Y with $X \subseteq Y$ and* $\dim Y/X = 1$ *such that Y has only trivial isometries.*

Observe that if $\dim Y/X < \infty$, then any norm on Y that preserves the norm on X is equivalent to the given norm on Y, so we get the following corollary.

COROLLARY. *For any Banach space X there is an equivalent norm $|||\cdot|||$ on X such that the space $(X, |||\cdot|||)$ has only trivial isometries.*

We divide the proof of Theorem 1 into a few steps.

PROPOSITION 1. *Let Γ be a set and E be a Banach space such that $c_0(\Gamma) \subseteq E \subseteq l^\infty(\Gamma)$. Then there is a norm $|||\cdot|||$ on E, equivalent with the original sup norm $\|\cdot\|$ of E and such that a linear map $T: E \to E$ is both $\|\cdot\|$- and $|||\cdot|||$-isometry if and only if $T = \lambda \,\mathrm{Id}_E$ with $|\lambda| = 1$.*

PROOF. Let $T: E \to E$ be a $\|\cdot\|$-isometry. Observe that elements e', e'' of ∂E_1 do *not* have disjoint supports if and only if

$$\exists\, e \in E_1 \quad \exists\, \alpha, \beta \in \partial K_1 \qquad \| e' + \alpha e'' + \beta e \| > 1$$

and

$$\forall \lambda \in K_1 \qquad \| e' + \lambda e \| \leqq 1, \quad \| e'' + \lambda e \| \leqq 1.$$

The above property is described only by linear and metric properties of E so it is preserved by T. Hence T maps elements of E with disjoint supports onto elements with disjoint supports. By standard arguments it follows now that T is of the form

$$T(e_\gamma) = \varepsilon_\gamma e_{\pi(\gamma)} \qquad \forall \gamma \in \Gamma$$

where $\pi: \Gamma \to \Gamma$ is a permutation and $|\varepsilon_\gamma| = 1$ for $\gamma \in \Gamma$.

We fix a well order $<$ on Γ and define for $x \in E$

$$|||x||| = \max\{ \|x\|, \sup\{ |2x(\gamma) + x(\beta)| : \gamma < \beta \in \Gamma\}\}.$$

Assume now that T is also a $|||\cdot|||$-isometry. To end the proof we have first to show that $\pi = \mathrm{Id}_\Gamma$ and then that $\varepsilon_\gamma = \varepsilon_{\gamma'}$ for any $\gamma, \gamma' \in \Gamma$. To prove that $\pi = \mathrm{Id}_\Gamma$ it is enough to show that it preserves the order, so assume by contradiction that $\gamma < \gamma'$, yet $\pi(\gamma) > \pi(\gamma')$. Then $|||2e_\gamma + e_{\gamma'}||| = 5$ but

$$|||T(2e_\gamma + e_{\gamma'})||| = |||2\varepsilon_\gamma e_{\pi(\gamma)} + \varepsilon_{\gamma'} e_{\pi(\gamma')}||| = \max\{2, |2\varepsilon_{\gamma'} + 2\varepsilon_\gamma|\} \leqq 4,$$

a contradiction. Finally if $\varepsilon_\gamma \neq \varepsilon_{\gamma'}$, then $|||e_\gamma + e_{\gamma'}||| = 3$, but

$$|||T(e_\gamma + e_{\gamma'})||| = |||\varepsilon_\gamma e_\gamma + \varepsilon_{\gamma'} e_{\gamma'}||| = \max\{2, |2\varepsilon_\gamma + \varepsilon_{\gamma'}|\} < 3.$$

Hence $\varepsilon_\gamma = \varepsilon_{\gamma'}$ for all $\gamma, \gamma' \in \Gamma$.

PROPOSITION 2. *For any Banach space X there is a set Γ and a continuous,* *linear injective map J from X into $l^\infty(\Gamma)$ such that the closure of $J(X)$ contains $c_0(\Gamma)$.*

The proposition is an immediate consequence and in fact is equivalent to a theorem of Plička [8] (see also [9], p. 861/862) that any Banach space has a biorthogonal, bounded and total system.

PROPOSITION 3. *Let $(X, \|\cdot\|)$ be a Banach space, x_0 a non-zero element of X, $p(\cdot)$ a continuous norm on $(X, \|\cdot\|)$, G_1 the group of all isometries of $(X, \|\cdot\|)$ and G_2 the group of all isometries T of $(X, p(\cdot))$ such that Tx_0 and x_0 are linearly dependent. Then there is a norm $\|\cdot\|_w$ on $Y = X \oplus K$ such that $\|\cdot\|_w$ and $\|\cdot\|$ coincide on X and the group G of all isometries of $(Y, \|\cdot\|_w)$ is isomorphic to $G_1 \cap G_2$.*

PROOF. Put $p'(\cdot) = p(\cdot) + \|\cdot\|$. Observe that the norm $p'(\cdot)$ is equivalent to the original norm $\|\cdot\|$ of X and that a linear map $T: X \to X$ preserves both $\|\cdot\|$ and $p'(\cdot)$ if and only if it preserves $\|\cdot\|$ and $p(\cdot)$. Hence without loss of generality we can assume that the norms $p(\cdot)$ and $\|\cdot\|$ are equivalent; we can also assume by multiplying by an appropriate positive number that

$$1000\|x\| \leqq p(x) \qquad \text{for } x \in X$$

and that

$$\|x_0\| \leqq 0.1.$$

We put

$$A = \{(x, \alpha) \in X \oplus K = Y : \max\{\|x\|, |\alpha|\} \leqq 1\},$$

$$C = \{(x + x_0, 2) : p(x) \leqq 1\},$$

and let $\|\cdot\|_w$ be the norm whose unit ball W is the closed balanced convex set generated by $A \cup C$.

Observe that

$$\|(x, \alpha)\|_w = \|x\| \qquad \forall (x, \alpha) \in Y, \quad |\alpha| \leqq \|x\|.$$

Hence the $\|\cdot\|_w$ norm coincides with the original one on X. It is also evident that if $T: X \to X$ preserves both norms $\|\cdot\|$ and $p(\cdot)$ and $Tx_0 = \lambda x_0$, $|\lambda| = 1$, then $T \oplus \mathrm{Id}_K$ is an isometry of Y.

Assume now that $T: Y \to Y$ is a $\|\cdot\|_w$-isometry. We prove the proposition by showing that there is a λ, $|\lambda| = 1$ so that

(i) T maps X onto X;

(ii) $T_{|X}$ preserves both $\|\cdot\|$ and $p(\cdot)$;

(iii) $T(x_0, 0) = (\lambda x_0, 0)$ and $T((0, 1)) = (0, \lambda)$ where $|\lambda| = 1$.

We note that C as well as all its rotations λC, $|\lambda| = 1$ are faces of W.

We distinguish two types of points in ∂W:

(1°) points interior to a segment I contained in ∂W, whose length (with respect to the W norm) is at least 0.1, and the limits of such points;

(2°) all other points.

As these types are metrically defined, they are preserved by T. On the other hand it is easy to see that the points of type (1°) cover all of ∂W except the relative interiors of the faces λC. Hence $T(x_0, 2) \in \lambda C$ with $|\lambda| = 1$. Replacing T by $\bar{\lambda} T$ we can assume that $T(x_0, 2) \in C$ and since T maps the face C onto a face of W we have $TC = C$. To prove now that T maps X onto X let $x \in X$, with $p(x) \leqq 1$. We have

$$T(x, 0) = T((x + x_0, 2) - (x_0, 2)) = T(x + x_0, 2) - T(x_0, 2) \in C - C \subset X,$$

and as $\{x : p(x) \leqq 1\}$ contains a ball in X this is true for all $x \in X$, i.e. $TX \subseteq X$; by symmetry $TX = X$. Because the $\|\cdot\|_w$ norm agrees with $\|\cdot\|$ on X, it follows that $T_{|X}$ is a $\|\cdot\|$-isometry.

Since $TC = C$, the function $T_{|X}$ maps $B := \{x \in X : p(x) \leqq 1\}$ onto itself. Hence for any $x \in X$ with $p(x) \leqq 1$ we have the following implications:

$$x_0 \pm x \in B \Rightarrow Tx_0 \pm Tx \in B \Rightarrow p((Tx_0 - x_0) \pm Tx) \leqq 1 \Rightarrow$$

$$p(Tx) \leqq \tfrac{1}{2}(p(Tx + (Tx_0 - x_0)) + p(Tx - (Tx_0 - x_0))) \leqq 1.$$

By symmetry we get $p(x) = p(Tx)$, and evidently $Tx_0 = x_0$, hence also $T(0, 1) = T(0, 1)$.

Now to end the proof of Theorem 1 let $(X, \|\cdot\|)$ be Banach space. Put $Y = X \oplus K$. We construct a norm on Y which coincides with the original norm of X on $X \cong X \oplus \{0\} \subset X \oplus K = Y$ and such that Y has only trivial isometries.

Let $J : X \to l^\infty(\Gamma)$ be an injective map given by Proposition 2 and let $|||\cdot|||$ be a norm on $E := \overline{J(x)} \subseteq l^\infty(\Gamma)$ given by Proposition 1. Fix $\gamma \in \Gamma$. We then have

$$E \cong \{e \in E : e(\gamma) = 0\} \oplus_\infty K,$$

so by Proposition 3 and 1 there is a continuous norm $\tilde{p}$ on E such that $(E, \tilde{p})$ has only trivial isometries. We define a continuous norm p on X by

$$p(x) = \tilde{p}(J(x)), \qquad x \in X.$$

Evidently $(J(X), \tilde{p}(\cdot))$ and so $(X, p(\cdot))$ have only trivial isometries. Hence, again by Proposition 3, there is a norm on $Y = X \oplus K$ with trivial isometries, which coincides with $\|\cdot\|$ on X.

§3. Remarks

As we have mentioned in the introduction it is an open problem to describe all groups that can be represented in a given Banach space, even in the separable Hilbert space.

Answering a question of Lindenstrauss, Gordon and Loewy [2] proved that if G is a finite group of operators on $\mathbf{R}^n$ containing $-I$, then there is a norm $|||\cdot|||$ on $\mathbf{R}^n$ such that G is the group of isometries of $(\mathbf{R}^n, |||\cdot|||)$. We conjecture that any group of the form $G\times\{-1,+1\}$ (or $G \times$ *the circle group*, respectively in the complex case) can be represented in a Banach space E provided $\mathrm{card}(E)\geqq \mathrm{card}(G)$ ($\mathrm{card}(E)>\mathrm{card}(G)$?). Proposition 3 and the following theorems give some partial information.

THEOREM 2. *For any group G there is a compact, connected Hausdorff space S such that the group of all isometries of $C_{\mathbf{R}}(S)$ and $G\times\{-1,1\}$ are isomorphic.*

PROOF. The theorem is an immediate consequence of the classical Banach–Stone theorem which says that any isometry T of $C_{\mathbf{R}}(S)$ is of the form

$$Tf = u\cdot f\circ\varphi \qquad \text{for } f\in C_{\mathbf{R}}(S),$$

where $\varphi: S\to S$ is a homeomorphism and u is a unimodular continuous function, and a result of J. De Groot [3] that for any group G there is a compact, connected space S such that G and the group of all homeomorphisms of S are isomorphic.

THEOREM 3. *Let S be a compact, metric, uncountable space. Then for any countable group G, $G\times\{-1,+1\}$ is representable in $C_{\mathbf{R}}(S)$.*

PROOF. The theorem follows immediately from the Banach–Stone theorem and from the following two results:

THEOREM [7]. *Let S_1, S_2 be compact, metric and uncountable spaces. Then Banach spaces $C_{\mathbf{R}}(S_1)$ and $C_{\mathbf{R}}(S_2)$ are isomorphic.*

THEOREM [4]. *For any countable group G there is a compact, connected subset S of the plane such that the group G is isomorphic with the group of homeomorphisms of S.*

In the complex case the proof of Theorem 2 does not work, since we have a great number of unimodular functions u, not just 2 like in the real case. However a similar result holds.

THEOREM 4. *For any group G there is a metric connected space S such that the group of all isometries of* Lip(S) *is isomorphic to the product of G and the multiplicative group of the unit circle.*

By Lip(S) we denote the Banach space of all complex Lipschitz, bounded functions on S with the norm defined by

$$\| f \| = \max \left\{ \sup_{s \in S} |f(s)|, \sup_{s_1 \neq s_2 \in S} \frac{|f(s_1) - f(s_2)|}{d(s_1, s_2)} \right\}.$$

PROOF. We use the notation and the methods of [5].

By the theorem of V. Kannan and M. Rajagopalan [6] there is a metric, connected set S such that the group of all isometries of S is isomorphic to G. Let K be the minimal compactification of S such that any $f \in \mathrm{Lip}(S)$ can be extended to a continuous function on K (such K is an obvious quotient space of the Čech–Stone compactification of S). For each $s_0 \in S \subset K$ there is an element f of $\mathrm{Lip}(S) \subset C(K)$ such that

$$\| f \|_{\mathrm{Lip}(S)} = 1 = f(s_0) > |f(s)| \qquad \forall s \in K \setminus \{s_0\}$$

so each functional of the form $\lambda \delta_s$, $|\lambda| = 1$, $s \in S$ is an extreme point of $(\mathrm{Lip}(S))^*$ and hence Lip(S) is an M subspace of $C(K)$. Now, exactly as in Example 3 of [5], we get that any isometry T of Lip(S) onto itself is of the form

$$T(f)(k) = u(k) \cdot f \circ \varphi(k) \qquad \text{for } f \in \mathrm{Lip}(S), \quad k \in K \tag{1}$$

where $\varphi : K \to K$ is a homeomorphism and u is a complex-valued unimodular function.

Now it is easy to check that (1) defines an isometry if and only if u is a constant function of absolute value 1 and $\varphi_{|S}$ is an isometry.

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